Section 10.2 – Use Combinations and the Binomial Theorem

It is not always important to count all of the different orders that a group of objects can be arranged. A **combination** is a selection of *r* objects from a group of *n* objects where the *order is not important.*

Combinations of *n* Objects Taken *r* at a Time
The number of combinations of r objects taken from a group of n distinct objects is denoted by $_nC_r$ and is given by the formula:

$$_nC_r = \frac{n!}{(n-r)!\,r!}$$

Example 1
A standard deck of 52 playing cards has 4 suits with 13 different cards in each suit.

Standard 52-Card Deck

K ♠	K ♥	K ♦	K ♣
Q ♠	Q ♥	Q ♦	Q ♣
J ♠	J ♥	J ♦	J ♣
10 ♠	10 ♥	10 ♦	10 ♣
9 ♠	9 ♥	9 ♦	9 ♣
8 ♠	8 ♥	8 ♦	8 ♣
7 ♠	7 ♥	7 ♦	7 ♣
6 ♠	6 ♥	6 ♦	6 ♣
5 ♠	5 ♥	5 ♦	5 ♣
4 ♠	4 ♥	4 ♦	4 ♣
3 ♠	3 ♥	3 ♦	3 ♣
2 ♠	2 ♥	2 ♦	2 ♣
A ♠	A ♥	A ♦	A ♣

a. If the order in which the cards are dealt is not important, how many different 5-card hands are possible?

$$_{52}C_5 = \frac{52!}{(52-5)!\,5!} = \frac{52!}{47!\,5!} = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5!}$$

$$= \boxed{2{,}598{,}960}$$

b. In how many 5-card hands are all 5 cards of the same color?

$$_2C_1 \cdot {}_{26}C_5 = \frac{2!}{1!\,1!} \cdot \frac{26!}{21!\cdot 5!}$$

$\uparrow$ 1 of 2 colors $\uparrow$ 5 of 26 cards in that color

$$= 2 \cdot 65{,}780$$

$$= 131{,}560$$

Example 2
Parents have 10 books that they can read to their children this week. Five of the books are nonfiction and 5 are fiction.

n

a. If the order in which they read the books in not important, how many different sets of 4 books can they choose?

$\uparrow$ r

$$_{10}C_4 = \frac{10!}{6!\,4!} = \boxed{210}$$

b. In how many groups of 4 books are all the books either nonfiction or fiction?

$$_5C_4 + {}_5C_4 = \frac{5!}{1!\cdot 4!} + \frac{5!}{1!\cdot 4!}$$

$$= \boxed{10}$$

Multiple Events

- When finding the number of ways both an event A **and** an event B can occur, you need to **multiply**.
- When finding the number of ways that event A **or** event B can occur, you **add**.

Example 3

The Student Senate consists of 6 seniors, 5 juniors, 4 saphomores, and 3 freshmen.

a. How many different committees of exactly 2 seniors and 2 juniors can be chosen?

$${}_6C_2 \cdot {}_5C_2 = \frac{6!}{4!2!} \cdot \frac{5!}{3!2!}$$

$$= \boxed{150}$$

b. How many different committees of at most 4 students can be chosen?

$6+5+4+3 = 18$ students

$${}_{18}C_1 + {}_{18}C_2 + {}_{18}C_3 + {}_{18}C_4 = \boxed{4047}$$

Try:

The local movie rental store is having a special on new releases. The new releases consist of 12 comedies, 8 action, 7 drama, 5 suspense, and 9 family movies.

a. You want exactly 2 comedies and 3 family movies. How many different movie combinations can you rent?

$${}_{12}C_2 \cdot {}_9C_3 = 5{,}544$$

b. You can afford at most 2 movies. How ~~much~~ many movie combinations can you rent?

$${}_{41}C_1 + {}_{41}C_2 = 861$$

Subtracting Possibilities

Counting problems that involve phrases like 'at least' or 'at most' are sometimes easier to solve by subtracting possibilities you do not want from the total number of possibilities.

Example 4

During the school year, the girl's basketball team is scheduled to play 12 home games. You want to attend at least 3 of the games. How many different combinations of games can you attend?

$_{12}C_3 + _{12}C_4 + _{12}C_5 \cdots + _{12}C_{12}$

Instead...

Attend or not attend $\rightarrow$ $2^{12} - (_{12}C_0 + _{12}C_1 + _{12}C_2) = \boxed{4017}$

Example 5

You are going to toss 10 different coins. How many different ways will at least 4 of the coins show heads?

show heads or tails

$2^{10} - (_{10}C_0 + _{10}C_1 + _{10}C_2 + _{10}C_3) = \boxed{848}$

Many of the relationships among combinations can be seen in the array of numbers known as

Pascal's Triangle.

n = 0 (0th row)	$_0C_0$	1
n = 1 (1st row)	$_1C_0$ $_1C_1$	1 1
n = 2 (2nd row)	$_2C_0$ $_2C_1$ $_2C_2$	1 2 1
n = 3 (3rd row)	$_3C_0$ $_3C_1$ $_3C_2$ $_3C_3$	1 3 3 1
n = 4 (4th row)	$_4C_0$ $_4C_1$ $_4C_2$ $_4C_3$ $_4C_4$	1 4 6 4 1
n = 5 (5th row)	$_5C_0$ $_5C_1$ $_5C_2$ $_5C_3$ $_5C_4$ $_5C_5$	1 5 10 10 5 1
n = 6 (6th row)		

Example 6

From a collection of 7 baseball caps, you want to trade 3. Use Pascal's triangle to find the number of combinations of 3 caps that ~~can be~~ traded.

$\boxed{_7C_3 = 35}$

1 ← 0th row

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

7th row → 1 7 21 35 35 21 7 1

$_7C_0$ $_7C_3$

Try:
Out of 5 finalists, your class must choose 3 class representatives. Use Pascal's triangle to find the number of combinations of 3 students that can be chosen as representatives.

Binomial Expansions – We'll now explore the connection between Pascal's triangle and binomial expansions.

$(x + y)^0 = 1$

$(x + y)^1 = x + y$

$(x + y)^2 = x^2 + xy + xy + y^2$

$\boxed{x^2 + 2xy + y^2}$

$(x + y)^3 = (x+y)(x+y)(x+y)$

$(x^2 + 2xy + y^2)(x+y)$

$\boxed{x^3 + 3x^2y + 3xy^2 + y^3}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$(x + y)^4 =$

$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Binomial Theorem
For any positive integer n, the binomial expansion of $(a + b)^n$ is:

____

Notice that each term in the expansion of $(a + b)^n$ has the form ____________ where r is an integer from 0 to n.

Example 7
Expand $(x + y)^6$

$1 \cdot x^6 + 6x^5y + 15x^4y^2$
$+ 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1 \cdot y^6$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Example 8

Expand $(3x-2)^4$

$1\cdot(3x)^4(-2)^0 + 4\cdot(3x)^3(-2)^1 + 6\cdot(3x)^2(-2)^2 + 4\cdot(3x)^1(-2)^3 + 1\cdot(-2)^4$

$81x^4 - 216x^3$

1

1 1

1 2 1

1 3 3 1

$\boxed{1 \quad 4 \quad 6 \quad 4 \quad 1}$

Example 9

Expand $(5-2y)^3$

Example 10

a. Find the coefficient of x^5 in the expansion of $(2x-7)^9$

b. Find the coefficient of x^3y^4 in $(2x-y)^7$.

Try:

Expand $(6-c)^4$

Find the coefficient of x^8 in the expansion of $(3x-2)^{10}$