Section 6 – Solve Exponential and Logarithmic Equations

Exponential equations are equations in which variable expressions occur as exponents. To solve these equations, we use the property below.

Property of Equality for Exponential Equations
If b is a positive number other than 1, then $b^x = b^y$ if and only if $x = y$.

Example 1
Solve by equating exponents.

a.) $9^{2x} = 27^{x-1}$

$(3^2)^{2x} = (3^3)^{x-1}$

$3^{4x} = 3^{3x-3}$

$4x = 3x-3$

$\boxed{x = -3}$

b.) $100^{7x+1} = 1000^{3x-2}$

$(10^2)^{7x+1} = (10^3)^{3x-2}$

$10^{14x+2} = 10^{9x-6}$

$14x+2 = 9x-6$

$5x = -8$

$x = \boxed{-8/5}$

c.) $81^{3-x} = \left(\frac{1}{3}\right)^{5x-6}$

$(3^4)^{3-x} = (3^{-1})^{5x-6}$

$3^{12-4x} = 3^{-5x+6}$

$12-4x = -5x+6$

$\boxed{x = -6}$

Sometimes it's not convenient to write each side of an exponential equation using the same base. When this happens, you can take the logarithm of each side.

Example 2
Solve by taking the logarithm of both sides.

a.) $2^x = 5$

$\log(2^x) = \log 5$

$x \cdot \log(2) = \log 5$

$x = \frac{\log 5}{\log 2}$

$\boxed{x = 2.32}$

b.) $7^{9x} = 15$

$9x \cdot \log(7) = \log(15)$

$9x = \frac{\log(15)}{\log(7)}$

$9x = 1.39$

$\boxed{x = .15}$

c.) $4e^{-0.3x} - 7 = 13$

$4e^{-0.3x} = 20$

$e^{-.3x} = 5$

$\ln(e^{-.3x}) = \ln 5$

$-.3x = \ln 5$

$\boxed{x = -5.36}$

Example 3
Newton's Law of Cooling states that for a cooling substance with initial temperature T_0, the temperature T after t minutes can be modeled by $T = (T_0 - T_R)e^{-rt} + T_R$ where T_R is the surrounding temperature and r is the substance's cooling rate.

Suppose on a cold October afternoon, you decide to make yourself a steaming mug of hot chocolate at your friend's house in Windsor. Your hot chocolate has been heated to 90°C and is poured into a mug and placed on a table in a room with a temperature of 20°C. If r = 0.145 when time t is measured in minutes, how long will it take for the hot chocolate to cool to a temperature of 30°C?

$T_0 = 90$

$T_R = 20$

$r = .145$

$T = 30$

$T = (90-20)e^{-.145t} + 20$

$30 = (70)e^{-.145t} + 20$

$10 = 70e^{-.14t}$

$\ln(\frac{1}{7}) = \ln(e^{-.145t})$

$\ln(\frac{1}{7}) = -.145t$

$t = \frac{\ln(\frac{1}{7})}{-.145}$

$\boxed{t = 13.4}$

Logarithmic equations are equations that involve logarithms of variable expressions. To solve these equations, we use the property below:

Property of Equality for Logarithmic Equations
If b, x, and y are positive numbers with $b \neq 1$, then $\log_b x = \log_b y$ if and only if $x = y$.

Example 4
Solve the equation.

a.) $\log_4(2x + 8) = \log_4(6x - 12)$

$2x + 8 = 6x - 12$

$20 = 4x$

$\boxed{x = 5}$

b.) $\ln(7x - 4) = \ln(2x + 11)$

$7x - 4 = 2x + 11$

$5x = 15$

$\boxed{x = 3}$

Example 5
Exponentiate to solve.

$\log_{\text{base}} \text{answer} = \text{exponent}$

a.) $\log_4(5x - 1) = 3$

$4^3 = 5x - 1$

$64 = 5x - 1$

$65 = 5x$

$\boxed{x = 13}$

b.) $\log_7(3x - 2) = 2$

$7^2 = 3x - 2$

$49 = 3x - 2$

$51 = 3x$

$\boxed{x = 17}$

c.) $\log 2x + \log(x - 5) = 2$

$\log(2x(x-5)) = 2$

$\log_{10}(2x^2 - 10x) = 2$

$10^2 = 2x^2 - 10x$

$100 = 2x^2 - 10x$

$0 = 2x^2 - 10x - 100$

$0 = x^2 - 5x - 50$

$0 = (x+5)(x-10)$

$x = \cancel{-5}, 10$

d.) $\log_6 3x + \log_6(x - 4) = 2$

$\log_6(3x(x-4)) = 2$

$\log_6(3x^2 - 12x) = 2$

$6^2 = 3x^2 - 12x$

$0 = 3x^2 - 12x - 36$

$0 = x^2 - 4x - 12$

$0 = (x-6)(x+2)$

$x = 6, \cancel{-2}$